ON THE FREQUENCY OF SMALL FRACTIONAL PARTS IN CERTAIN REAL SEQUENCES

BY

WILLIAM J. LEVEQUE

1. **Introduction.** Let $X_1, X_2, \cdots$ be a sequence of independent random variables, each uniformly distributed on $[0, 1/2]$. If f is an arbitrary function from the positive integers to $[0, 1/2]$, the equation

$$\Pr\{X_k < f(k)\} = 2f(k) \tag{1}$$

holds, and it is a consequence of the Borel-Cantelli lemmas [3] that the probability that the inequality $X_k < f(k)$ is satisfied for infinitely many k is zero or one, according as the series

$$\sum_{k=1}^{\infty} f(k) \tag{2}$$

is convergent or divergent. While it is well known that no such general assertion can be made when the X_k are dependent, Khinchin [6] has found a direct analogue in an important case. His theorem is usually stated in measure-theoretic language: the inequality $|kx-p| < f(k)$ has infinitely many integral solutions k, p for almost all x or almost no x, according as (2) diverges or converges. We may, however, consider x as a random variable uniformly distributed over some interval, and define the quantity U_k $(k=1, 2, \cdots)$ as the distance $\langle kx\rangle$ between kx and the nearest integer to kx. Then the U_k form a sequence of dependent random variables uniformly distributed on $[0, 1/2]$; Khinchin's theorem shows that the nature of the dependence is not such as to affect the finiteness of the number of solutions of the inequality $U_k < f(k)$.

From a probabilistic standpoint the Borel-Cantelli lemmas yield very crude information about a sequence of random variables, and it is of some interest to know whether the U_k also resemble the X_k in their finer structure. We consider here the case in which (2) diverges, so that there are almost surely infinitely many solutions of $|kx-p| < f(k)$, and investigate in §§2–3 the number T_n of such solutions with $k \leqq n$. The result is not quite what would be expected from the case of independent variables. For if we put Y_k equal to 1 or 0 according as the inequality $X_k < f(k)$ does or does not hold, then $S_n = Y_1 + \cdots + Y_n$ is the number of $k \leqq n$ such that $X_k < f(k)$. Since

Presented to the Society, August 23, 1956; received by the editors September 17, 1956.

$$E(Y_k) = 1\cdot 2f(k) + 0\cdot(1 - 2f(k)) = 2f(k),$$

$$\operatorname{Var} Y_k = E(Y_k^2) - E^2(Y_k) = 2f(k) - 4f^2(k),$$

$$E(S_n) = 2\sum_{k=1}^{n} f(k),$$

$$\operatorname{Var} S_n = 2\sum_{k=1}^{n} f(k) - 4\sum_{k=1}^{n} f^2(k),$$

we deduce from the central limit theorem that if $\sum_1^\infty f^2(k)$ converges, then

$$\lim_{n\to\infty} \Pr\left\{S_n < 2\sum_{k=1}^{n} f(k) + \omega\left(2\sum_{k=1}^{n} f(k)\right)^{1/2}\right\} = \phi(\omega), \tag{3}$$

where

$$\phi(\omega) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\omega} e^{-u^2/2}du$$

is the normal distribution function.

The law of the iterated logarithm yields the closely related result that

$$\Pr\left\{\limsup_{n\to\infty}\left|\frac{S_n - 2\sum_{k=1}^{n} f(k)}{4\left(\sum_{k=1}^{n} f(k)\log\log\sum_{k=1}^{n} f(k)\right)^{1/2}}\right| = 1\right\} = 1$$

and so in particular

$$\Pr\left\{S_n \sim 2\sum_{k=1}^{n} f(k)\right\} = 1. \tag{4}$$

Theorem 1 exhibits the result corresponding to (3) for T_n; it differs from (3) in that the coefficient 2 is replaced by $12\pi^{-2}$.

In §§4–6 we consider the much less strongly dependent sequence $\langle r_1 r_2 \cdots r_k x\rangle$, where $r_1, r_2, \cdots$ is a fixed increasing sequence of positive integers, and show that here the situation is again as described in (3) and (4).

2. **A lemma.** Let f be a function with the following properties:

(5) $f(x)$ is positive and decreasing for $x \geqq 0$;

(6) $f(x) = O(x^{-1})$ and $f'(x) = O(x^{-2})$ as $x \to \infty$;

$$\sum_{k=1}^{\infty} f(k) = \infty. \tag{7}$$

We shall need some further properties of f, which we collect in the following lemma.

LEMMA 1. *If f satisfies (5)–(7) and if c and δ are positive constants, then*

(a) $$\sum_{k=1}^{n} f(k) = \int_1^n f(u)du + O(1);$$

(b) $$f(k + O(k^{1-\delta})) = f(k) + O(k^{-1-\delta});$$

(c) $$\sum_{k=1}^{cn} f(k) = \sum_{k=1}^{n} f(k) + O(1);$$

(d) $$\sum_{k=1}^{n} cf(ck) = \sum_{k=1}^{n} f(k) + O(1);$$

(e) $$\sum_{k=1}^{n} f(k) = c\sum_{k=1}^{e^n} \frac{f(c \log k)}{k} + O(1),$$

(f) *if $a_1, a_2, \cdots$ and α are such that*

$$\sum_{k=1}^{n} a_k \sim n\alpha$$

as $n \to \infty$, then

$$\sum_{k=1}^{n} a_k f(k) = \alpha \sum_{k=1}^{n} f(k) + O(1).$$

Part (a) is trivial, and (b) follows from (6) and the law of the mean. Part (c) follows from the estimate

$$\sum_{k=n}^{cn} f(k) = \sum_{k=n}^{cn} O(k^{-1}) = O(\log cn - \log n) = O(1),$$

and (d) from the fact that

$$\sum_{k=1}^{n} cf(ck) = \int_1^n cf(cu)du + O(1) = \int_c^{cn} f(t)dt + O(1) = \sum_{k=c}^{cn} f(k) + O(1).$$

The substitution $u = c \log v$ in (a) gives (e). To obtain (f), write

$$\sum_{k=1}^{n} (a_k - \alpha)f(k) = f(n)\sum_{k=1}^{n} (a_k - \alpha) + \sum_{k=1}^{n-1}\left(\sum_{l=1}^{k} (a_l - \alpha)\right)(f(k) - f(k+1))$$

and note that

$$f(n)\sum_{k=1}^{n} (a_k - \alpha) = O(n^{-1})o(n) = o(1)$$

and

$$\sum_{k=1}^{n-1}\left(\sum_{l=1}^{k}(a_l-\alpha)\right)(f(k)-f(k+1)) = \sum_{k=1}^{n-1} o(k)(f(k)-f(k+1))$$
$$= O(n)\sum_{k=1}^{n-1}(f(k)-f(k+1))$$
$$= O(nf(n)) = O(1).$$

We shall use the following notation: $\mathfrak{M}\{A\}$ means the measure of the set of $x\in[0, 1]$ such that A, if A is a sentence, and it means the measure of A if A is a set.

$\text{No}\{m\leqq n|\ \cdots\}$ means the number of positive integers $m\leqq n$ such that . . . .

$E_x\{\ \cdots\}$ or $\{x|\ \cdots\}$ means the set of $x\in[0, 1]$ such that $\cdots$.

3. **The fractional part of** mx. We prove the following theorem:

THEOREM 1. *Suppose that f satisfies conditions* (5)–(7) *and put*

$$g(x) = f(\log x)/x.$$

Let

$$T_n = T_n(x) = \text{No}\{m \leqq n \mid \langle mx\rangle < g(m)\}.$$

Then for fixed ω,

$$\lim_{n\to\infty} \mathfrak{M}\left\{T_n < \frac{12}{\pi^2}\sum_{k=1}^{n} g(k) + \omega\left(\frac{12}{\pi^2}\sum_{k=1}^{n} g(k)\right)^{1/2}\right\} = \phi(\omega).$$

If x is a real number with continued fraction expansion

$$x = a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots = a_0 + \frac{1}{a_1+}\cdots\frac{1}{a_k+}\,\frac{1}{x_{k+1}}$$

and convergents

$$\frac{p_k}{q_k} = a_0 + \frac{1}{a_1+}\cdots\frac{1}{a_k},$$

then

$$x = \frac{p_k x_{k+1} + p_{k-1}}{q_k x_{k+1} + q_{k-1}}$$

and

$$|q_k x - p_k| = \frac{1}{q_k x_{k+1} + q_{k-1}}.$$

LEMMA 2. *Put*

$$W_n = \text{No}\left\{k \leqq n \mid \ |q_k x - p_k| < \frac{f(k)}{q_k}\right\}.$$

Then

$$\lim_{n\to\infty} \mathfrak{M}\left\{W_n < \frac{1}{\log 2}\sum_{k=1}^{n} f(k) + \omega\left(\frac{1}{\log 2}\sum_{k=1}^{n} f(k)\right)^{1/2}\right\} = \phi(\omega).$$

We take x as a random variable uniformly distributed on $[0, 1]$, and use Pr_k, E_k and Var_k to denote conditional probability, expectation and variance when $a_0, \cdots, a_k$ are given. We suppose throughout this section that f satisfies conditions (5)–(7), and we put $\alpha_k = f(k)(1+q_{k-1}/q_k)$ and

$$V_k = \begin{cases} 1 - \alpha_k \ \text{if} \ |q_k x - p_k| < \dfrac{f(k)}{q_k}, \\ -\alpha_k \quad \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \mathrm{Pr}_k\{V_k = 1 - \alpha_k\} &= \mathrm{Pr}_k\left\{\frac{1}{(q_k x_{k+1} + q_{k-1})} < \frac{f(k)}{q_k}\right\} \\ &= \mathrm{Pr}_k\left\{x_{k+1} > \frac{1}{f(k)} - \frac{q_{k-1}}{q_k}\right\} \\ &= \mathrm{Pr}_k\left\{x \in \left[\frac{p_k(1/f(k) - q_{k-1}/q_k) + p_{k-1}}{q_k(1/f(k) - q_{k-1}/q_k) + q_{k-1}}, \frac{p_k}{q_k}\right]\right\} \\ &= \frac{\left|\dfrac{p_k q_k/f(k) \pm 1}{q_k^2/f(k)} - \dfrac{p_k}{q_k}\right|}{\left|\dfrac{p_k + p_{k-1}}{q_k + q_{k-1}} - \dfrac{p_k}{q_k}\right|} \\ &= f(k)\left(1 + \frac{q_{k-1}}{q_k}\right) = \alpha_k. \end{aligned}$$

Hence

$$\begin{aligned} E_k(V_k) &= (1 - \alpha_k)\alpha_k + (-\alpha_k)(1 - \alpha_k) = 0, \\ \mu_k^2 &= E_k(V_k^2) = f(k)\left(1 + \frac{q_{k-1}}{q_k}\right) + O(f^2(k)). \end{aligned} \tag{8}$$

P. Lévy [9; 10, p. 321] has shown that

$$\Pr\left\{\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{q_{k-1}}{q_k}\right)\sim\frac{1}{\log 2}\right\}=1,$$

and it follows from (f) of Lemma 1 that for almost all x,

$$\sum_{k=1}^{n} f(k)\left(1+\frac{q_{k-1}}{q_k}\right)=\frac{1}{\log 2}\sum_{k=1}^{n} f(k)+O(1). \tag{9}$$

Combining (8) and (9), we see that for almost all x,

$$\mu_1^2+\cdots+\mu_n^2=\frac{1}{\log 2}\sum_{k=1}^{n} f(k)+O(1). \tag{10}$$

We now use a form of the central limit theorem for dependent variables due to Lévy [10, p. 246] (and later extended by J. L. Doob [2, p. 383] as a theorem on martingales):

LEMMA 3. *Let $Z_1, Z_2, \cdots$ be a sequence of bounded random variables, and let E_{n-1} denote conditional expectation for given $Z_1, \cdots, Z_{n-1}$. Suppose that $E_{n-1}(Z_n)=0$ for $n\geqq 2$, and put*

$$\mu_n^2=E_{n-1}(Z_n^2)=\mathrm{Var}_{n-1}(Z_n).$$

For $t>0$, determine $N=N(t)$ so that

$$\mu_1^2+\cdots+\mu_N^2\sim t,$$

and put

$$S(t)=Z_1+\cdots+Z_N.$$

Then if

$$\Pr\left\{\sum_{n=1}^{\infty}\mu_n^2<\infty\right\}=0,$$

we have

$$\lim_{t\to\infty}\Pr\left\{\frac{S(t)}{t^{1/2}}<\omega\right\}=\phi(\omega).$$

If $Z_k=V_k$, it follows from (10) that aside from a set of measure 0, the functions $N(t)$ corresponding to various x's are asymptotically equal, and that

$$\lim_{n\to\infty}\Pr\left\{\frac{V_1+\cdots+V_n}{\left(\frac{1}{\log 2}\sum_{k=1}^{n} f(k)\right)^{1/2}}<\omega\right\}=\phi(\omega).$$

But

$$W_n = \sum_{k=1}^{n} V_k + \sum_{k=1}^{n} f(k)\left(1 + \frac{q_{k-1}}{q_k}\right),$$

and hence for almost all x,

$$W_n = \sum_{k=1}^{n} V_k + \frac{1}{\log 2}\sum_{k=1}^{n} f(k) + O(1).$$

Thus

$$\lim_{n\to\infty} \Pr\left\{W_n < \frac{1}{\log 2}\sum_{k=1}^{n} f(k) + \omega\left(\frac{1}{\log 2}\sum_{k=1}^{n} f(k)\right)^{1/2}\right\} = \phi(\omega), \tag{11}$$

which completes the proof of the lemma.

The remainder of the proof of Theorem 1 consists in transforming (11) into a statement not involving continued fractions. For this we need an estimate of q_k.

Lemma 4. *If $\delta < 1/2$, then for almost every x there is a constant $\kappa = \kappa(x, \delta)$ such that*

$$\left|\log q_k - \frac{\pi^2}{12 \log 2} k\right| < \kappa k^{1-\delta}.$$

This results from an extension of the following theorem of Khinchin [7]: *Let F be a function of k positive integral arguments, such that for $n \geqq k$,*

$$\int_0^1 F^2(a_n, \cdots, a_{n-k+1})dx < C,$$

where $a_m = a_m(x)$ denotes the mth denominator in the continued fraction expansion of x. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=k}^{n} F(a_i, \cdots, a_{i-k+1})$$

exists and is constant almost everywhere.

Examination of the proof shows that the theorem may be modified in two ways. The function F may be replaced by a quantity depending on a slowly increasing number of the a_m; we write

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F_i(a_i, a_{i-1}, \cdots, a_{i-k_i+1}), \tag{12}$$

and require that $i - k_i + 1$ be positive for $i \geqq 1$. Secondly, the rapidity of approach of the sum in (12) to its limiting value can be estimated by replacing the ϵ occurring in Khinchin's proof by $n^{-\epsilon}$, where ϵ is now a sufficiently small positive constant. In this way the following theorem can be proved:

Let $\{F_i(r_1, \cdots, r_{k_i})\}$ be non-negative functions of the positive integral arguments $r_1, r_2, \cdots$, and suppose that the integrals

$$\int_0^1 F_i^2(a_i, a_{i-1}, \cdots, a_{i-k_i+1})dx$$

are uniformly bounded. Suppose further that $\delta < 1/2$ and that

$$k_i = O(\log^\sigma i)$$

for some constant $\sigma > 0$. Then there is a constant B such that

$$\frac{1}{n}\sum_{i=1}^{n} F_i(a_i, \cdots, a_{i-k_i+1}) = B + O(n^{-\delta})$$

for almost all x.

We put

$$\phi_i(x) = a_i + \frac{1}{a_{i-1}+} \cdots \frac{1}{a_{i-k_i+1}}$$

and

$$F_i(a_i, \cdots, a_{i-k_i+1}) = \log \phi_i(x).$$

Since $\phi_i(x) \leqq a_i + 1$ and $\mathfrak{M}\{a_i = r\} = \mathfrak{M}\{r \leqq x_i < r+1\} < 1/r^2$, we have

$$\int_0^1 F_i^2 dx \leqq \int_0^1 \log^2 (a_i + 1)dx \leqq \sum_{r=1}^{\infty} \frac{\log^2 (r+1)}{r^2} .$$

Thus for

$$k_i = 1 + [2 \log i]$$

there is a B_0 such that for almost all x,

$$\sum_{i=1}^{n} \log \phi_i(x) = B_0 n + O(n^{1-\delta}).$$

On the other hand, if $\bar{\phi}_i(x) = q_i/q_{i-1}$, then by the law of the mean,

$$\left| \log \phi_i(x) - \log \bar{\phi}_i(x) \right| = \xi \left| \phi_i(x) - \bar{\phi}_i(x) \right|$$

where $\xi < 1$. Since

$$\bar{\phi}_i(x) = a_i + \frac{1}{a_{i-1}+} \cdots \frac{1}{a_1}, \tag{13}$$

this implies that

$$\left| \log \phi_i(x) - \log \bar\Phi_i(x) \right| < \left| \left(a_i + \frac{1}{a_{i-1}+} \cdots \frac{1}{a_{i-k_i+1}} \right) - \left(a_i + \frac{1}{a_{i-1}+} \cdots \frac{1}{a_{i-k_i+1}+1} \right) \right| < 1/Q_{k_i}^2,$$

where P_l/Q_l is the lth convergent in the expansion (13). Since

$$Q_l \geqq Q_{l-1} + Q_{l-2} > 2Q_{l-2} > \cdots > 2^{[l/2]},$$

we see that

$$\left| \log \phi_i(x) - \log \bar\Phi_i(x) \right| < 2^{1-k_i} < i^{-2 \log 2}.$$

Thus for almost all x,

$$\sum_{i=1}^{n} \log \bar\Phi_i(x) = \log q_n = B_0 n + O(n^{1-\delta}).$$

Lévy [10, p. 320] showed that $B_0 = \pi^2/12 \log 2$. The proof of Lemma 4 is complete.

Now let

$$s_n = \mathrm{No}\left\{ k \leqq n \,\middle|\, |q_k x - p_k| < \frac{f(B_0^{-1} \log q_k)}{q_k} \right\},$$

$$t_n(\kappa) = \mathrm{No}\left\{ k \leqq n \,\middle|\, |q_k x - p_k| < \frac{f(k - \kappa k^{1-\delta})}{q_k} \right\}.$$

By (11),

$$\lim_{n\to\infty} \mathfrak{M}\left\{ t_n(\kappa) < \frac{1}{\log 2} \sum_{k=1}^{n} f(k - \kappa k^{1-\delta}) + \omega \left(\frac{1}{\log 2} \sum_{k=1}^{n} f(k - \kappa k^{1-\delta}) \right)^{1/2} \right\} = \phi(\omega).$$

Putting

$$A_n = \frac{1}{\log 2} \sum_{k=1}^{n} f(k),$$

it follows from (b) of Lemma 1 that for each κ,

$$\lim_{n\to\infty} \mathfrak{M}\{ t_n(\kappa) < A_n + \omega A_n^{1/2} \} = \phi(\omega). \tag{14}$$

Let

$$F_n = \{ x \mid s_n < A_n + \omega A_n^{1/2} \},$$

$$G(\kappa) = \{ x \mid \, |\log q_k - B_0 k| < \kappa k^{1-\delta} \text{ for every } k \geqq 1 \},$$

$$H_n(\kappa) = \{ x \mid t_n(\kappa) < A_n + \omega A_n^{1/2} \}.$$

Then by Lemma 2 and Equation (14), to each $\epsilon>0$ there corresponds a $\kappa_0=\kappa_0(\epsilon)$ and an $n_0=n_0(\kappa_0, \epsilon)=n_0(\epsilon)$ such that

$$\mathfrak{M}\{G(\kappa)\} > 1 - \epsilon \qquad \text{for } \kappa \geqq \kappa_0$$

and

$$\left| \mathfrak{M}\{H_n(\pm \kappa_0)\} - \phi(\omega) \right| < \epsilon \qquad \text{for } n \geqq n_0.$$

Clearly

$$G(\kappa_0)H_n(\kappa_0) \subset F_n,$$

and since $\mathfrak{M}(AB) \geqq \mathfrak{M}(A)+\mathfrak{M}(B)-1$ if A and B are subsets of $[0, 1]$, we have that for $n \geqq n_0$,

$$\mathfrak{M}\{F_n\} \geqq 1 - \epsilon + \phi(\omega) - \epsilon - 1 = \phi(\omega) - 2\epsilon.$$

Similarly, since $G(\kappa_0)F_n \subset H_n(-\kappa_0)$,

$$\mathfrak{M}\{F_n\} \leqq \phi(\omega) + 2\epsilon.$$

Hence

$$\lim_{n\to\infty} \mathfrak{M}\{F_n\} = \lim_{n\to\infty} \Pr\left\{s_n < A_n + \omega A_n^{1/2}\right\} = \phi(\omega).$$

By the same reasoning we can use (d) of Lemma 1 to show that if

$$r_n = \text{No}\left\{k \leqq n \,\middle|\, |q_k x - p_k| < \frac{B_0 f(\log q_k)}{q_k}\right\},$$

then

$$\lim_{n\to\infty} \Pr\left\{r_n < A_n + \omega A_n^{1/2}\right\} = \phi(\omega).$$

Replacing f by f/B_0, it follows immediately that

$$\lim_{n\to\infty} \Pr\left\{\text{No}\left\{k \leqq n \,\middle|\, |q_k x - p_k| < \frac{f(\log q_k)}{q_k}\right\} < \frac{12}{\pi^2}\sum_{k=1}^{n} f(k) + \omega\left(\frac{12}{\pi^2}\sum_{k=1}^{n} f(k)\right)^{1/2}\right\} = \phi(\omega).$$

If $|mx-l| < 1/2m$, then l/m is a convergent to x. Since $f(x)=o(1)$,

$$\text{No}\left\{k \leqq n \,\middle|\, |q_k x - p_k| < \frac{f(\log q_k)}{q_k}\right\} = \text{No}\left\{m \leqq q_n \,\middle|\, \langle mx\rangle < \frac{f(\log m)}{m}\right\} + O(1),$$

the error term being uniformly bounded for all x. Putting

$$A(n) = \frac{12}{\pi^2} \sum_{k=1}^{e^n} \frac{f(\log k)}{k}$$

and using (e) of Lemma 1 with $c=1$, it follows that

$$\lim_{n\to\infty} \Pr \left\{ \text{No} \left\{ m \leqq q_n \,\middle|\, \langle mx\rangle < \frac{f(\log m)}{m} \right\} < A(n) + \omega A(n)^{1/2} \right\} = \phi(\omega). \tag{15}$$

There is now a final set-theoretic argument required to eliminate q_n entirely. Put

$$F(n, \omega) = E_x \left\{ \text{No} \left\{ m \leqq q_n \,\middle|\, \langle mx\rangle < \frac{f(\log m)}{m} \right\} < A(n) + \omega A(n)^{1/2} \right\},$$

$$G(n, \beta, \omega) = E_x \left\{ \text{No} \left\{ m \leqq e^{\beta n} \,\middle|\, \langle mx\rangle < \frac{f(\log m)}{m} \right\} < A(n) + \omega A(n)^{1/2} \right\},$$

$$H_N(\epsilon) = E_x \left\{ e^{B_0(1-\epsilon)\nu} < q_\nu < e^{B_0(1+\epsilon)\nu} \text{ for all } \nu \geqq N \right\}.$$

It is easily seen that

$$H_N(\epsilon)G(n, B_0(1+\epsilon), \omega) \subset F(n, \omega), \quad H_N(\epsilon)F(n, \omega) \subset H_N(\epsilon)G(n, B_0(1-\epsilon), \omega) \tag{16}$$

for $0<\epsilon<1$, $n \geqq N$. On the other hand, we have

$$G\left(\frac{1-\epsilon}{1+\epsilon}n, B_0(1+\epsilon), \eta\right) = E_x \left\{ \text{No} \left\{ m \leqq e^{B_0(1-\epsilon)n} \,\middle|\, \langle mx\rangle < \frac{f(\log m)}{m} \right\} \right.$$
$$\left. < A\left(\frac{1-\epsilon}{1+\epsilon}n\right) + \eta A^{1/2}\left(\frac{1-\epsilon}{1+\epsilon}n\right) \right\},$$

and hence if η is chosen so that

$$A\left(\frac{1-\epsilon}{1+\epsilon}n\right) + \eta A^{1/2}\left(\frac{1-\epsilon}{1+\epsilon}n\right) > A(n) + \omega A^{1/2}(n), \tag{17}$$

then

$$G\left(\frac{1-\epsilon}{1+\epsilon}n, B_0(1+\epsilon), \eta\right) \supset G(n, B_0(1-\epsilon), \omega). \tag{18}$$

By (c) of Lemma 1, $A(cn) = A(n) + O(1)$, so

$$A\left(\frac{1-\epsilon}{1+\epsilon}n\right) + \eta A^{1/2}\left(\frac{1-\epsilon}{1+\epsilon}n\right) = A(n) + (\eta + O(A^{-1/2}(n)))A^{1/2}(n).$$

Since $A(n)\to\infty$ as $n\to\infty$, it follows that if $\delta>0$ is arbitrary, (17) holds with $\eta = \omega+\delta$, if $n > n_0(\epsilon, \delta)$. But then by (16) and (18),

$$H_N(\epsilon)F(n, \omega) \subset H_N(\epsilon)G(n, B_0(1-\epsilon), \omega) \subset F\left(\frac{1-\epsilon}{1+\epsilon}n, \omega+\delta\right)$$

for

$$n > \min\left(\frac{1+\epsilon}{1-\epsilon}N, n_0\right).$$

By Lemma 4, $\mathfrak{M}\{H_N(\epsilon)\}\to 1$ as $N\to\infty$, and by (15), $\mathfrak{M}\{F(n, \omega)\}\to\phi(\omega)$ as $n\to\infty$. Hence, if we allow n and N to increase in such a way that

$$N(1+\epsilon)/(1-\epsilon) < n,$$

we obtain the inequality

$$\phi(\omega) \leqq \lim_{n\to\infty} \mathfrak{M}\{G(n, B_0(1-\epsilon), \omega)\} \leqq \phi(\omega+\delta).$$

Since δ is arbitrary and ϕ is continuous,

$$\lim_{n\to\infty} \mathfrak{M}\{G(n, B_0(1-\epsilon), \omega)\} = \phi(\omega).$$

Since ϵ is arbitrary (in $[0, 1]$), we can choose $\epsilon = 1 - B_0^{-1}$, and obtain

$$\lim_{n\to\infty} \mathfrak{M}\{G(n, 1, \omega)\} = \phi(\omega),$$

or

$$\lim_{n\to\infty} \Pr\left\{\mathrm{No}\left\{m \leqq e^n \,\middle|\, \langle mx\rangle < \frac{f(\log m)}{m}\right\}\right. \\ \left. < \frac{12}{\pi^2}\sum_{k=1}^{e^n}\frac{f(\log k)}{k} + \omega\left(\frac{12}{\pi^2}\sum_{k=1}^{e^n}\frac{f(\log k)}{k}\right)^{1/2}\right\} = \phi(\omega).$$

Using (c) of Lemma 1 again (with $1 \leqq c \leqq (n+1)/n$) and the fact that there are at most three denominators q_k lying between e^n and e^{n+1}, we obtain Theorem 1.

4. **The small values of** $\langle r_1 r_2 \cdots r_n x\rangle$. We now consider sequences of the form $\langle r_1 r_2 \cdots r_n x\rangle$, where x is again uniformly distributed on $[0, 1]$ and $r_1, r_2, \cdots$ is a fixed nondecreasing sequence of integers larger than 1, not depending on x, with $\lim r_n = \infty$. Let the sequences $\{x_n\}$ and $\{a_n\}$ of real numbers and integers, respectively, be determined by the following conditions:

$$\begin{aligned} r_1 x &= a_1 + x_1, & -1/2 &\leqq x_1 < 1/2, \\ r_2 x_1 &= a_2 + x_2, & -1/2 &\leqq x_2 < 1/2, \\ &\cdots\cdots & & \\ r_n x_{n-1} &= a_n + x_n, & -1/2 &\leqq x_n < 1/2, \\ &\cdots\cdots & & \end{aligned}$$

Then

$$a_n = [r_n x_{n-1} + 1/2], \qquad |x_n| = \langle r_n x_{n-1}\rangle, \tag{19}$$

$$-\left[\frac{r_n}{2}\right] \leqq a_n < \left[\frac{r_n}{2}\right], \tag{20}$$

for $n=1, 2, \cdots$, and

$$x = \sum_{n=1}^{\infty} \frac{a_n}{r_1 \cdots r_n}. \tag{21}$$

The series (21) bears an obvious relation to the expansion of x to the base r if, contrary to assumption, we take all $r_n = r$, and to the Cantor factorial expansion if $r_n = n$ for all n. In any case, the expansion is unique except for a set of measure zero.

Since x is a random variable, so is every element of $\{x_n\}$ and $\{a_n\}$, and it is easily seen that each x_n is uniformly distributed on $[-1/2, 1/2]$, and that each a_n is discretely uniformly distributed, in the sense that

$$\Pr\{a_n = j\} = \frac{1}{r_n} \quad \text{for} \quad -\left[\frac{r_n}{2}\right] \leqq j < \left[\frac{r_n}{2}\right]. \tag{22}$$

There is a significant difference between the two sets of variables, however, in that the a_n are statistically independent, while the x_n are not, as the Equations (19) show. Dependence makes the sequence $\{x_n\}$ difficult to analyze probabilistically, but a considerable amount of information can be gained indirectly by transferring results about $\{a_n\}$ via the relation

$$x_{n-1} = \frac{a_n}{r_n} + O\left(\frac{1}{r_n}\right).$$

THEOREM 2. *Suppose that $r_1, r_2, \cdots$ is a nondecreasing sequence of positive integers such that $r_n^m > n$ for some fixed integer m. Let $R_n = r_1 r_2 \cdots r_n$, and let f be a positive function. Let S be an increasing sequence of positive integers. Then the inequality*

$$\langle R_n x\rangle < f(n) \tag{23}$$

has infinitely many solutions $n \in S$ for almost all x or almost no x, according as the series

$$\sum_{n\in S} f(n) \tag{24}$$

diverges or converges.

We note first that it suffices to consider functions f such that $f(n) \geqq n^{-2}$ for all $n \in S$. For if (24) converges, then so does the series

$$\sum_{n\in S} f^*(n),$$

where

$$f^*(n) = \begin{cases} f(n) & \text{if } f(n) \geqq n^{-2}, \\ n^{-2} & \text{otherwise,} \end{cases}$$

and if the inequality $\langle R_n x\rangle < f^*(n)$ has only finitely many solutions in S, the same is surely true of (23). Suppose on the other hand that (24) diverges. Then so also does

$$\sum f(n_j),$$

the summation being extended over the integers $n_j \in S$ such that $f(n_j) \geqq n_j^{-2}$. These integers constitute a subsequence S' of S, and the truth of the theorem for S' implies its truth for S.

We suppose throughout the proof that $n\in S$. If we put

$$P_n = R_n \sum_{j=1}^{n} \frac{a_j}{R_j},$$

then

$$| R_n x - P_n | = | x_n | \leqq 1/2,$$

so

$$| R_n x - P_n | = \langle R_n x\rangle.$$

For each n let k_n be the unique positive integer such that

$$(25) \qquad [r_{n+1} \cdots r_{n+k_n-1} f(n) + 1/2] = 0, \quad [r_{n+1} \cdots r_{n+k_n} f(n) + 1/2] \neq 0;$$

in particular, if $[r_{n+1}f(n)+1/2] \neq 0$ then $k_n = 1$. Then

$$(26) \qquad \frac{1}{r_{n+1} \cdots r_{n+k_n}} \leqq 2f(n).$$

Let $\mathcal{E}_n$ be the event that (i.e., the set of $x\in[0, 1]$ such that)

$$a_{n+1} = \cdots = a_{n+k_n-1} = 0, \quad | a_{n+k_n} | < r_{n+1} \cdots r_{n+k_n} f(n) + 1 = \frac{R_{n+k_n}}{R_n} f(n) + 1,$$

and for $c>0$ let $\mathcal{F}_n(c)$ be the event that $\langle R_n x\rangle < cf(n)$.

Suppose that $x\in\mathcal{F}_n(1)$. If $k_n=1$, then we have

$$| x_n | < f(n),$$

$$| a_{n+1} | = | a_{n+k_n} | = | [r_{n+1}x_n + 1/2] | \leqq r_{n+1} | x_n | + 1/2 < r_{n+k_n} f(n) + 1,$$

so $x\in\mathcal{E}_n$. If $k_n > 1$, then

$$
\begin{aligned}
|a_{n+1}| &\leqq [r_{n+1}f(n) + 1/2] = 0, \qquad x_{n+1} = r_{n+1}x, \\
|a_{n+2}| &\leqq [r_{n+1}r_{n+2}f(n) + 1/2] = 0, \qquad x_{n+2} = r_{n+1}r_{n+2}x_n, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
|a_{n+k_n-1}| &\leqq [r_{n+1}\cdots r_{n+k_n-1}f(n) + 1/2] = 0, \quad x_{n+k_n-1} = r_{n+1}\cdots r_{n+k_n-1}x_n, \\
|a_{n+k_n}| &\leqq [r_{n+1}\cdots r_{n+k_n}f(n) + 1/2] < r_{n+1}\cdots r_{n+k_n}f(n) + 1,
\end{aligned}
$$

and again $x \in \mathcal{E}_n$. Hence $\mathcal{F}_n(1) \subset \mathcal{E}_n$.

On the other hand, if $x \in \mathcal{E}_n$ then

$$x = \sum_{j=1}^{n} \frac{a_j}{R_j} + \sum_{j=n+k_n}^{\infty} \frac{a_j}{R_j},$$

so

$$
\begin{aligned}
\langle R_n x\rangle &= |R_n x - P_n| < \frac{(|a_{n+k_n}| + 1)R_n}{R_{n+k_n}} < \frac{\left(\dfrac{R_{n+k_n}}{R_n}f(n) + 2\right)R_n}{R_{n+k_n}} \\
&= f(n) + \frac{2R_n}{R_{n+k_n}}
\end{aligned}
\tag{27}
$$

and it follows from (26) that $\mathcal{E}_n \subset \mathcal{F}_n(3)$.

Thus if $\mathcal{E}_n$ occurs for only finitely many $n \in S$, the same is true of $\mathcal{F}_n(1)$; while if $\mathcal{E}_n$ occurs for infinitely many $n \in S$, the same is true of $\mathcal{F}_n(3)$. Since the convergence of (24) is unaffected by replacing $f(n)$ by $3f(n)$, there remains only the task of showing that $\mathcal{E}_n$ occurs for infinitely many $n \in S$, or only finitely many $n \in S$, for almost all x, according as (24) diverges or converges.

Since $r_n^m > n$ and $f(n) > n^{-2}$, we have $r_{n+1} \cdots r_{n+2m}f(n) > 1$. Hence $k_n \leqq 2m$, and the event $\mathcal{E}_n$ depends on at most the $2m$ random variables $a_{n+1}, \cdots, a_{n+2m}$. Hence for fixed l $(0 \leqq l < 2m)$, the events $\mathcal{E}_{2\nu m+l}$ $(\nu = 0, 1, \cdots)$ are independent. By (22),

$$
\Pr\{|a_n| = j\} = \begin{cases} \dfrac{1}{r_n} & \text{if } j = 0, \\[2ex] \dfrac{2}{r_n} & \text{if } 0 < j < \dfrac{r_n}{2}, \\[2ex] \dfrac{1}{r_n} & \text{if } j = \dfrac{r_n}{2}, \quad r_n \text{ even.} \end{cases}
$$

Hence for arbitrary real $u \in [0, r_n/2)$,

$$
\Pr\{|a_n| \leqq u\} = \frac{2[u] + 1}{r_n} \begin{cases} \leqq (2u + 1)/r_n, \\ \geqq (2u - 1)/r_n. \end{cases}
$$

Thus, because of the independence of the a_n, we have

$$\Pr\{\mathcal{E}_n\} \leqq \frac{1}{r_{n+1}} \cdots \frac{1}{r_{n+k_n-1}} \cdot \frac{2\dfrac{R_{n+k_n}}{R_n} f(n) + 3}{r_{n+k_n}} = 2f(n) + \frac{3R_n}{R_{n+k_n}}, \tag{28}$$

and by (26),

$$\Pr\{\mathcal{E}_n\} < 8f(n).$$

Also

$$\Pr\{\mathcal{E}_n\} \geqq \frac{2\dfrac{R_{n+k_n}}{R_n} f(n) + 1}{r_{n+1} \cdots r_{n+k_n}} > 2f(n). \tag{29}$$

Hence for each l the series[1]

$$\sum_{\nu; 2\nu m + l \in S} \Pr\{\mathcal{E}_{2\nu m + l}\}$$

converges or diverges with the series

$$\sum_{\nu; 2\nu m + l \in S} f(2\nu m + l). \tag{30}$$

But if the series

$$\sum_{n \in S} f(n) \tag{31}$$

diverges, at least one of the series (30), for $0 \leqq l < 2m$, must diverge, while if (31) converges, all the series (30) converge. The theorem therefore follows from the Borel-Cantelli lemmas.

5. We now consider the case in which (23) has infinitely many solutions for almost all x, and investigate the number of such solutions with $n \leqq N$. For simplicity we suppose that S is the full set of positive integers.

THEOREM 3. *Let* $\{r_n\}$ *and* $\{R_n\}$ *be as described in Theorem* 2. *Let* f *be a positive function such that*

$$\sum_{n=1}^{\infty} f(n) = \infty, \qquad f(n) = O(n^{-1/2-\epsilon}).$$

Let k_n *be the positive integer defined in* (25), *and suppose that*

$$\sum_{n=1}^{\infty} (r_{n+1} \cdots r_{n+k_n})^{-1} < \infty. \tag{32}$$

---

(1) The symbol $\sum_{\nu;} \cdots$ means summation over those ν such that $\cdots$.

Then

$$\lim_{N\to\infty} \Pr \left\{ \mathrm{No}\left\{ n \leqq N \mid \langle R_n x\rangle < f(n) \right\} < 2\sum_{n=1}^{N} f(n) + \omega\left(2\sum_{n=1}^{N} f(n)\right)^{1/2} \right\} = \phi(\omega). \tag{33}$$

According to Theorem 2, the n for which $f(n) < n^{-2}$ contribute only a bounded number of solutions of the inequality (23), so we may suppose that $f(n) \geqq n^{-2}$. Put

$$X_n = \begin{cases} 1 & \text{if } \langle R_n x\rangle < f(n), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$S_N = \sum_{n=1}^{N} X_n.$$

Similarly, put

$$Y_n = \begin{cases} 1 & \text{if } \mathcal{E}_n \text{ occurs,} \\ 0 & \text{otherwise} \end{cases}$$

and

$$T_N = \sum_{n=1}^{N} Y_n,$$

where $\mathcal{E}_n$ has the same meaning as before. Since $\mathfrak{F}_n(1) \subset \mathcal{E}_n$, we have

$$S_N < T_N. \tag{34}$$

On the other hand, if $Y_n = 1$ then either $X_n = 1$ or

$$\langle R_n x\rangle \in \left[f(n), f(n) + \frac{2R_n}{R_{n+k_n}} \right], \tag{35}$$

by (27). Because of the uniform distribution of the x_n, the probability of the event (35) is $2R_n/R_{n+k_n}$, and by (32) and the first Borel-Cantelli lemma, the event (35) occurs only finitely many times, for almost all x. Thus given $\epsilon > 0$, there is a constant M so large that

$$T_N < S_N + M \tag{36}$$

for all N and all x not in a set of measure at most ϵ. Combining (34) and (36), we see that (33) will follow if it can be shown that

$$\lim_{N\to\infty} \Pr \left\{ T_N < 2\sum_{n=1}^{N} f(n) + \omega\left(2\sum_{n=1}^{N} f(n)\right)^{1/2} \right\} = \phi(\omega). \tag{37}$$

To this end we first prove a general lemma, suggested by work of Hoeffding and Robbins [5]. A set of random variables $Z_1, Z_2, \cdots$ is said to be m-dependent if for every r,s and n for which $n>s>r+m$, the sets $Z_1, \cdots, Z_r$ and $Z_s, \cdots, Z_n$ are independent. (The variables Y_n above are $2m$-dependent.)

THEOREM 4. *Let $Z_1, Z_2, \cdots$ be a sequence of m-dependent random variables such that*

$$Z_n = \begin{cases} 1 \text{ with probability } p_n, \\ 0 \text{ with probability } 1 - p_n. \end{cases}$$

Suppose that

$$\sum_{n=1}^{\infty} p_n = \infty, \tag{38}$$

$$p_n = O(n^{-1/2-\epsilon}), \qquad \epsilon > 0, \tag{39}$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left| \operatorname{Cov}(Z_i, Z_{i+j}) \right| < \infty. \tag{40}$$

Then

$$\lim_{n\to\infty} \Pr \left\{ Z_1 + \cdots + Z_n < \sum_{k=1}^{n} p_k + \omega\left(\sum_{k=1}^{n} p_k\right)^{1/2} \right\} = \phi(\omega).$$

We decompose the finite sequence $1, 2, \cdots, n$ into blocks, in the following way. Choose η smaller than ϵ, and find an integer l_0 such that

$$(l_0 + 1)^{2+\eta} - l_0^{2+\eta} > 2m. \tag{41}$$

For $q \geqq 1$ put

$$l_q = [(l_0 + q)^{2+\eta}],$$

and define $\kappa = \kappa(n)$ by the inequality

$$l_\kappa \leqq n < l_{\kappa+1}.$$

For $1 \leqq q < \kappa - 1$, let I_{q+1} be the set of integers j such that $l_q < j \leqq l_{q+1} - m$, and let J_{q+1} be the set of integers j such that $l_{q+1} - m < j \leqq l_{q+1}$. Finally, put

$$U_q = \sum_{\nu \in I_q} Z_\nu = \sum_{I_q} Z_\nu,$$

$$V_q = \sum_{J_q} Z_\nu,$$

for $q=2, \cdots, \kappa$, so that

$$Q_n = \sum_{\nu=1}^{n} Z_\nu = \sum_{\nu=1}^{l_1} Z_\nu + \sum_{q=2}^{\kappa} U_q + \sum_{q=2}^{\kappa} V_q + \sum_{\nu=l_\kappa+1}^{n} Z_\nu.$$

By the definitions of l_0 and m-dependence, the variables $U_2, \cdots, U_\kappa$ are independent, as are $V_2, \cdots, V_\kappa$. We shall show that the limiting behavior of Q_n is determined by that of $\sum U_q$, and then apply a standard version of the central limit theorem.

Since l_1 is fixed and the Z's are bounded, the sum

$$\sum_{\nu=1}^{l_1} Z_\nu$$

is clearly negligible in the limit, if $\mathrm{Var}\,(S_n) \to \infty$. By (40), (39), and (38),

$$\begin{aligned}
\mathrm{Var}\left(\sum_{q=2}^{\kappa} V_q\right) &= \sum_{q=2}^{\kappa} \sum_{J_q} \mathrm{Var}\,(Z_\nu) + 2 \sum_{q=2}^{\kappa} \sum_{J_q} \mathrm{Cov}\,(Z_\mu, Z_\nu) \\
&= \sum_{q=2}^{\kappa} \sum_{J_q} (p_\nu - p_\nu^2) + O(1) \\
&= \sum_{q=2}^{\kappa} \sum_{J_q} p_\nu + O(1) \\
&= \sum_{q=2}^{\kappa} \sum_{\nu=1}^{m} O(l_q^{-1/2-\epsilon}) + O(1) \\
&= O\left(\sum_{q=2}^{\kappa} q^{-1-2\epsilon-\eta/2-\epsilon\eta} \right) + O(1),
\end{aligned}$$

so that

$$\mathrm{Var}\left(\sum_{q=2}^{\kappa} V_q\right) = O(1). \tag{42}$$

Turning to U_q, we see that

$$E(U_q) = \sum_{I_q} p_\nu = e_q, \tag{43}$$

and

$$\mathrm{Var}\,(U_q) = \sum_{I} \mathrm{Var}\,(Z_\nu) + 2 \sum_{\mu,\nu \in I_q;\, \mu<\nu<\mu+m} \mathrm{Cov}\,(Z_\mu, Z_\nu),$$

so that

$$\sigma_\kappa^2 = \mathrm{Var}\,(U_2 + \cdots + U_\kappa) = \sum_{q=2}^{\kappa} e_q + O(1). \tag{44}$$

Now

$$e_q < c \sum_{I_q} \frac{1}{\nu^{1/2+\epsilon}} < c(l_q^{1/2-\epsilon} - l_{q-1}^{1/2-\epsilon})$$
$$< c l_{q-1}^{1/2-\epsilon} \left\{ \left(1 + \frac{1}{q}\right)^{(1/2-\epsilon)(2+\eta)} - 1 \right\}$$
$$= O\left(q^{(2+\eta)(1/2-\epsilon)} \cdot \frac{1}{q} \right)$$

and so

$$e_q = O(1). \tag{45}$$

This implies in particular that

$$\operatorname{Var}\left(\sum_{\nu=l_\kappa+1}^{n} Z_\nu \right) = O(1), \tag{46}$$

and hence, since

$$\sum_{q=2}^{\infty} \sum_{J_q} p_\nu < \infty,$$

that

$$\sigma_\kappa^2 = \sum_{\nu=1}^{n} p_\nu + O(1), \qquad E(U_2 + \cdots + U_\kappa) = \sum_{\nu=1}^{n} p_\nu + O(1). \tag{47}$$

If we put

$$\pi_n = \sum_{\nu=1}^{n} p_\nu,$$

then (42) shows that

$$\operatorname{Var}\left(\pi_n^{-1/2} \sum_{q=2}^{\kappa} V_q \right) = O(1),$$

and it follows from Chebyshev's inequality that the random variable $\pi_n^{-1/2} \sum_2^\kappa V_q$ approaches zero in probability. By the same reasoning this is true also of $\pi_n^{-1/2} \sum_1^{l_1} Z_\nu$. Combining these facts with (46), we see [1, p. 254] that the limiting distribution of $(Q_n - \pi_n)/\pi_n^{1/2}$ is identical with that of

$$(U_2 + \cdots + U_\kappa - \pi_n)/\pi_n^{1/2}. \tag{48}$$

We now wish to apply Lyapunov's criterion [1, p. 213], according to which the normalized sum (48) is asymptotically normal, with mean zero and variance 1, if

$$\left(\sum_{q=2}^{\kappa} \rho_q^3\right)^{1/3} = O(\sigma_\kappa), \tag{49}$$

where

$$\rho_q^3 = E(|\,U_q - E(U_q)\,|^3).$$

This will complete the proof of Theorem 4. We have

$$\begin{aligned}\rho_q^3 &\leqq E\left\{\left(\sum_{I_q} |\,Z_\nu - p_\nu\,|\right)^3\right\} \\ &< 6E\left\{\sum_{\nu\in I_q} |\,Z_\nu - p_\nu\,|^3 + \sum_{\mu,\nu\in I_q} |\,Z_\mu - p_\mu\,| \cdot |\,Z_\nu - p_\nu\,|^2 \right.\\ &\qquad \left. + \sum_{\mu,\nu,\lambda\in I_q} |\,Z_\mu - p_\mu\,| \cdot |\,Z_\nu - p_\nu\,| \cdot |\,Z_\lambda - p_\lambda\,|\right\}.\end{aligned}$$

Now

$$\begin{aligned}\sum_{I_q} E(|\,Z_\nu - p_\nu\,|^3) &= \sum_{I_q} (1 - p_\nu)^3 p_\nu + \sum_{I_q} p_\nu^3(1 - p_\nu) \\ &= e_q + O\left(\sum_{I_q} p_\nu^2\right).\end{aligned}$$

Since $|Z_\nu - p_\nu| < 1$, we have, by the generalized Hölder inequality [4, p. 140],

$$\begin{aligned}\sum_{\mu,\nu\in I_q} E(|\,Z_\mu - p_\mu\,| \cdot |\,Z_\nu - p_\nu\,|^2) &\leqq \sum_{\mu,\nu\in I_q} E(|\,Z_\mu - p_\mu\,| \cdot |\,Z_\nu - p_\nu\,|) \\ \leqq \left(\sum_{\mu,\nu\in I_q} \operatorname{Var}(Z_\mu) \operatorname{Var}(Z_\nu)\right)^{1/2} &\leqq \sum_{\mu\in I_q} \operatorname{Var}(Z_\mu) \\ = \sum_{\mu\in I_q} (p_\mu - p_\mu^2) &= e_q + O\left(\sum_{I_q} p_\mu^2\right).\end{aligned}$$

Similarly,

$$\begin{aligned}&\sum_{\mu,\nu,\lambda\in I_q} E(|\,Z_\mu - p_\mu\,| \cdot |\,Z_\nu - p_\nu\,| \cdot |\,Z_\lambda - p_\lambda\,|) \\ &\qquad \leqq \left\{\sum_{I_q} E(|\,Z_\mu - p_\mu\,|^3) E(|\,Z_\nu - p_\nu\,|^3) E(|\,Z_\lambda - p_\lambda\,|^3)\right\}^{1/3} \\ &\qquad \leqq \sum_{I_q} E(|\,Z_\mu - p_\mu\,|^3) \qquad = e_q + O\left(\sum_{I_q} p_\mu^2\right).\end{aligned}$$

Thus (49) reduces to the triviality

$$\sum_{q=2}^{\kappa} e_q + O(1) = o\left\{\left(\sum_{q=2}^{\kappa} e_q\right)^{3/2}\right\}.$$

To complete the proof of Theorem 3, we must show that the hypotheses of Theorem 4 are satisfied when $Z_n = Y_n$, $p_n = \Pr\{\mathcal{E}_n\}$. We know that

$$2f(n) \leqq p_n \leqq 8f(n),$$

and hence, from the hypotheses of Theorem 3, we obtain (38) and (39). Since the Y_n are $2m$-dependent, we can rewrite (40) in the form

$$\sum_{i=1}^{\infty} \sum_{j=1}^{2m} \left| \operatorname{Cov}(Y_i, Y_{i+j}) \right| < \infty. \tag{50}$$

Now if $j > k_n$, then Y_i and Y_{i+j} are independent, and their covariance is 0. If $i \leqq j \leqq k_n$, then

$$\begin{aligned} \left| \operatorname{Cov}(Y_i, Y_{i+j}) \right| &= \left| E(Y_i Y_{i+j}) - E(Y_i)E(Y_{i+j}) \right| \\ &= \left| \Pr\{Y_i = Y_{i+j} = 1\} - \Pr\{Y_i = 1\} \cdot \Pr\{Y_{i+j} = 1\} \right| \\ &\leqq (r_{n+1} \cdots r_{n+k_n})^{-1} + 8f(i)f(i+j), \end{aligned}$$

and the convergence of (50) follows from (32).

6. **A strong theorem.**

THEOREM 5. *Let $\{R_n\}$ and $f(n)$ satisfy the hypotheses of Theorem 3. Then for almost all x, the number of integers $m \leqq n$, for which $\langle R_m x\rangle < f(m)$, is asymptotic to*

$$2 \sum_{k=1}^{n} f(k).$$

As in the proof of Theorem 3, it suffices to prove the theorem with S_n replaced by $T_n = \sum_1^n Y_k$, and to suppose that $f(n) > n^{-2}$, so that the Y_k are $2m$-dependent. We write

$$\begin{aligned} T_n &= \sum\nolimits^* Y_{2m\nu+1} + \sum\nolimits^* Y_{2m\nu+2} + \cdots + \sum\nolimits^* Y_{2m\nu+2m} \\ &= T_n^{(1)} + T_n^{(2)} + \cdots + T_n^{(2m)}, \end{aligned}$$

where each summation extends over those ν for which the subscripts are not larger than n. The terms in $T_n^{(j)}$ are independent and uniformly bounded, and

$$E(T_n^{(j)}) = 2 \sum\nolimits^* f(2m\nu + j), \qquad \operatorname{Var}(T^{(j)}) = 2 \sum\nolimits^* f(2m\nu + j) + O(1).$$

Hence Kolmogorov's version of the law of the iterated logarithm [8] implies that for $1 \leqq j \leqq 2m$,

$$\Pr\left\{ \limsup_{n \to \infty} \frac{\left| T_n^{(j)} - 2\sum^* f(2m\nu + j) \right|}{2(\sum^* f(2m\nu + j) \cdot \log\log \sum^* f(2m\nu + j))^{1/2}} = 1 \right\} = 1;$$

and it follows from these equations that

$$\Pr\left\{\left|T_n - 2\sum_{k=1}^{n} f(k)\right| = O\left(\sum_{j=1}^{2m}\left(\sum{}^{*} f(2m\nu + j)\cdot\log_2 \sum{}^{*} f(2m\nu + j)\right)^{1/2}\right)\right\} = 1,$$

and the theorem is a weak consequence of this result.

Note added in proof.

I. There is a strong version of Theorem 1:

Under the hypotheses of Theorem 1, *the number of solutions* $m \leqq n$ *of the inequality* $\langle mx\rangle < g(m)$ *is asymptotic to*

$$\frac{12}{\pi^2}\sum_{k=1}^{n} g(k),$$

for almost all x.

The proof depends on a strong law of large numbers for dependent variables, due to Lévy [10, p. 253]: *Under the hypotheses of Lemma* 3,

$$\Pr\left\{\lim_{t\to\infty}\frac{S(t)}{t^{1/2+\epsilon}} = 0\right\} = 1$$

for every positive constant ϵ. Using this in place of Lemma 3, we obtain a strong analogue of Lemma 2, to the effect that for $\epsilon > 0$,

$$\Pr\left\{W_n - (\log 2)^{-1}\sum_{1}^{n} f(k) = o\left(\left(\sum_{1}^{n} f(k)\right)^{1/2+\epsilon}\right)\right\} = 1,$$

and thereafter the proof parallels that of Theorem 1.

II. It has been pointed out to me that Lemma 3 is not immediately applicable in the proof of Lemma 2, since $E_k(V_k)$, in the equation preceding (8), means $E(V_k$, given $a_0, \cdots, a_k)$ and not $E(V_k$, given $V_0, \cdots, V_{k-1})$, and it is possible that V_{k-1}, for example, is not uniquely determined by $a_0, \cdots, a_k$. But in order for this to be the case it is necessary, since $|q_{k-1}x - p_{k-1}| = (q_{k-1}x + q_{k-2})^{-1}$ and $a_k = [x_k]$, that

$$\frac{1}{q_{k-1}(a_k + 1) + q_{k-2}} < \frac{f(k-1)}{q_{k-1}} < \frac{1}{q_{k-1}a_k + q_{k-2}}\,.$$

This happens only if

$$a_k = \left[\frac{1}{f(k-1)} - \frac{q_{k-2}}{q_{k-1}}\right].$$

The difficulty vanishes, therefore, if we prove the following theorem, and exclude from the beginning the exceptional set mentioned in it (taking $b = 1$ and $h(k) = 1/f(k-1)$):

Let h *be a real-valued function on the positive integers, with* $h(k) > ck$ *for some positive constant* c. *Then for every positive constant* b, *the set of* x, *for which the inequality* $|a_k - h(k)| < b$ *has infinitely many solutions, has measure zero.*

Put $F_k(t) = \Pr\{x_k < t\}$; then Lévy's form of the Gauss-Kuzmin theorem [10, pp. 298–306] asserts that for some g with $0 < g < 1$,

$$\left| F_k(t) - \frac{1}{\log 2} \log \frac{2t}{t+1} \right| < g^{k-1}$$

for all $t > 1$ and all positive integers k. Now the inequality $|a_k - h(k)| < b$ is equivalent to

$$h(k) - b < x_k < h(k) + b + 1,$$

and we have

$$\begin{aligned} \lim_{k\to\infty} \Pr\{h(k) - b < x_k < h(k) + b + 1\} & \\ < \frac{1}{\log 2} \log \left(\frac{2(h(k) + b + 1)}{h(k) + b + 2} \cdot \frac{h(k) - b + 1}{2(h(k) - b)} \right) + 2g^{k-1} & \\ = \frac{1}{\log 2} \log \frac{2h^2(k) + 4h(k) - 2(b^2 - 1)}{2h^2(k) + 4h(k) - 2(b^2 + b)} + 2g^{k-1} & \\ = \frac{1}{\log 2} \log (1 + O(h^{-2}(k))) + 2g^{k-1} = O(h^{-2}(k)) + 2g^{k-1}. & \end{aligned}$$

Hence the probabilities of the inequalities in question form the terms of a convergent series, and the required result follows from the Borel-Cantelli lemma.

References

1. H. Cramér, *Methods of mathematical statistics*, Princeton University Press, 1946.
2. J. L. Doob, *Stochastic processes*, New York, J. Wiley and Sons, 1953.
3. W. Feller, *Probability theory and its applications*, New York, J. Wiley and Sons, 1950.
4. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, 1934.
5. W. Hoeffding and H. Robbins, *The central limit theorem for dependent random variables*, Trans. Amer. Math. Soc. vol. 15 (1948) pp. 773–780.
6. A. Ya. Khinchin, *Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der diophantischen Approximationen*, Math. Ann. vol. 92 (1924) pp. 115–125.
7. ———, *Zur metrischen Kettenbruchtheore*, Comp. Math. vol. 3 (1936) pp. 276–285.
8. A. Kolmogorov, *Über das Gesetz des iterierten Logarithmus*, Math. Ann. vol. 101 (1929) pp. 126–135.
9. P. Lévy, *Sur le développement en fraction continue d'un nombre choisie au hasard*, Comp. Math. vol. 3 (1936) pp. 286–303.
10. ———, *Théorie de l'addition des variables aléatoires*, Paris, Gauthier-Villars, 1937.

University of Michigan,
Ann Arbor, Mich.